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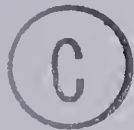
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THE UNIVERSITY OF ALBERTA

ISOLATORS IN POLYCYCLIC GROUPS

by



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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Isolators in Polycyclic Groups" submitted by Mohamed F. Yousif in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In this thesis we investigate the following class χ of groups: We say a group G is in χ if G is soluble of finite rank and there exists a subgroup G_0 of finite index in G and a finite set of primes π_1 such that whenever $\pi_1 \subseteq \pi$ (a set of primes) and $K \leq G_0$ then the set $\{g \in G_0: g^m \in K, \text{ for some } \pi\text{-number } m\}$ is a subgroup, called the π -isolator of K in G_0 denoted by $K_{\pi}^{G_0}$, and $|K_{\pi}^{G_0}: K|$ is a π -number.

In Chapter 1 a summary of known results concerning the theory of isolators developed by P. Hall [1] is presented. From his results it follows that if G is finitely generated nilpotent by finite group then G is always in χ .

In Chapter 2, the class of Fitting Isolated groups is defined and we give a proof to one of Rhemtulla and Hartley's fruitful results, that every torsion free soluble group of finite rank has a Fitting Isolated subgroup of finite index, this result is used in the investigation of the class χ .

Denote by $\sqrt[G]{H}$ to the set $\{g \in G: g^m \in H, m \geq 1\}$ where $H \leq G$. In Chapter 3 we show that, if G is not nilpotent, there is no reason to expect $\sqrt[G]{H}$ to be always a subgroup. However it has been proved by Rhemtulla and Wehrfritz that every polycyclic group G has a subgroup G_0 of finite index such that $\sqrt[G_0]{H}$ is a subgroup for every $H \leq G_0$.

In Chapter 4 the main result of this thesis is given. It is an example which shows that polycyclic metabelian groups already fail to be in χ . As a result of this example and a discussion with A. Rhemtulla we conjecture that G is in χ if and only if G is finitely generated nilpotent by finite group.

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Chapter 0

Notations and Definitions

If X and Y are groups, we write $X \leq Y$ to indicate that X is a subgroup of Y . $X < Y$ means that X is a proper subgroup of Y .

When X is a subset of a group, $\langle X \rangle$ will denote the subgroup generated by the elements of X . If X consists of the elements $x_1, x_2, \dots$ we also write $\langle x_1, x_2, \dots \rangle$.

For any subgroup H of G , $N_G(H)$ denotes the normalizer of H in G . $C_G(H)$ denotes the centraliser of H in G . For any integer $n \neq 0$,

$$G^n = \{g^n : g \in G\}.$$

The commutator $[a, b]$ of any two elements a and b of G is given by the equation

$$[a, b] = a^{-1}b^{-1}ab.$$

Let H and K be any subgroups of G . Then $[H, K]$ will denote the subgroup generated by the set of all commutators $[h, k]$ with $h \in H$ and $k \in K$.

If H, K are subgroups of G such that $H \leq K$, then $|H:K|$ denotes the index of H in K , i.e. the number of cosets of H in K .

If G is a group and we have a series of subgroups,
 $G = G_1 \geq G_2 \geq \dots \geq G_n \geq G_{n+1} = \{1\}$, then the series is a subnormal series if G_{i+1} is a normal subgroup of G_i , $1 \leq i \leq n$. The series is a normal series if each subgroup G_i is a normal subgroup of G .

A group G is soluble if it has a finite normal series,

$G = G_1 \geq G_2 \geq \dots \geq G_{n+1} = \{1\}$, such that G_i/G_{i+1} is Abelian ($1 \leq i \leq n$).

A group G is polycyclic if it has a finite subnormal series of subgroups $G = G_1 \geq G_2 \geq \dots \geq G_{n+1} = \{1\}$, such that G_i/G_{i+1} is cyclic ($1 \leq i \leq n$).

A group G is supersoluble if it has a finite normal series of subgroups $G = G_1 \geq G_2 \geq \dots \geq G_{n+1} = \{1\}$ such that, G_i/G_{i+1} is cyclic ($1 \leq i \leq n$).

A group G is nilpotent if it is soluble, and the normal series $\{G_i\}$ ($1 \leq i \leq n$) can be chosen so that the action of G on each of the factors G_i/G_{i+1} is trivial, i.e. every element of G acts as the identity automorphism. That is to say $[G_i, G] \leq G_{i+1}$ ($1 \leq i \leq n$) or equivalently, G_i/G_{i+1} is in the center of G/G_{i+1} . For this reason, any series satisfying this condition is called a central series of G .

A particular central series, defined for any group G , is the lower central series $\{\gamma_i\}$, defined as follows: $\gamma_1 = G$, and $\gamma_{i+1} = [\gamma_i, G]$. It is easy to show inductively that, G is nilpotent if and only if for some positive integer c , $\gamma_{c+1} = \{1\}$. The number c obtained in this way is called the class of nilpotency of the nilpotent group G .

Another central series, in some ways dual to the series γ_i , is the upper central series, $\{Z_i\}$, defined as follows: Z_1 is the center of G , and $Z_{i+1} = \{x \in G: xZ_i \text{ is in the center of } G/Z_i\}$. It is easy to verify that G is nilpotent if and only if for some positive integer n , $Z_n = G$. The smallest such integer is the class, c , defined above.

For a complete reference, we refer the reader to P. Hall [1] and

to B. Warfield [9].

Finally the following lemma will be used in the next chapter, for the proof see [1].

LEMMA (A): (i) A group G is polycyclic if and only if it is soluble and all its subgroups are finitely generated.

(ii) Finitely generated nilpotent groups are supersoluble.

(iii) Every supersoluble group G has a nilpotent subgroup K such that $|G:K| < \infty$ and $K \geq G'$, where $G' = [G, G]$.

Chapter 1

Isolators in Nilpotent Groups

We study here the theory of isolators developed by P. Hall [1] and list the main definitions and results that will be used in this thesis.

Definition 1.1: Let π be a set of prime numbers. A π -number is a positive integer whose prime divisors lie in π . A π -group is a finite group of order a π -number. If π contains just one prime p , we write p -number and p -group. Thus a p -group here always means a finite p -group.

Given a group G , an element $g \in G$ is called an m -element if its order divides a power of m . The π -elements of G are the elements whose orders are π -numbers.

Definition 1.2: Let P be a property of groups. A group G is called locally P if every finitely generated subgroup of G has the property P .

The following lemma will be useful to prove the main result of this Chapter, for the proof see [1, page 12].

Lemma 1.3: Let $G = \{a_1, \dots, a_r\}$ be a finitely generated nilpotent group. If $|G:HG'| = m$ is finite, where H is a subgroup of G , then $|G:H|$ is an m -number. In particular, if for each $i = 1, \dots, r$ we have $a_i^{m_i} \in H$ where m_i is a π -number, then $|G:H|$ is a π -number. If $|G:G'| = m$ is finite, then $|G|$ is an m -number. If $a_i^{m_i} = 1$, for $i = 1, 2, \dots, r$, where m_i is a π -number then $|G|$ is a π -number.

Definition 1.4: For any subgroup H of a group G , and any set of primes π , let H_{π}^G denote the set $\{g \in G: g^m \in H, \text{ for some } \pi\text{-number } m\}$.

We say H has a π -isolator in G , if H_{π}^G is a subgroup of G , and that H is π -isolated in G , if $H = H_{\pi}^G$.

Theorem 1.5: Let G be a locally nilpotent group, H any subgroup, π any set of primes. Then H_{π}^G is a subgroup, called the π -isolator of H in G . If G is finitely generated and therefore nilpotent, then $|H_{\pi}^G:H|$ is a π -number.

Proof: Let x^m, y^n belong to H , where $m \& n$ are π -numbers, and if $K = \{x, y\}$ and $L = H \cap K$, then $|K:LK'|$ is finite and divides mn . But K is finitely generated and therefore nilpotent. By Lemma 1.3, $|K:L|$ is a π -number, and in particular $(xy)^{\alpha} \in L \leq H$ for some π -number α . Thus H_{π}^G is a group.

If G is finitely generated, and therefore nilpotent, so is H_{π}^G by lemma A(i). Hence $|H_{\pi}^G:H(H_{\pi}^G)'|$ is a π -number. Hence $|H_{\pi}^G:H|$ is a π -number by 1.3.

As an immediate result of this theorem we can state the following Corollary.

Corollary 1.6: The π -elements of a locally nilpotent group G form a fully invariant subgroup of G . If G is finitely generated, the π -elements form a finite subgroup.

Our goal in this thesis is to try to extend the result of 1.5 to a bigger class of groups. And that will be done in Chapters 3 and 4.

The next theorem plays a good role in the next chapters, but

before that we need the following Lemma and for the proof see [1, page 12].

Lemma 1.7: Let $H = \{x, y\}$ be nilpotent and $x^m = y^m$, then xy^{-1} is an m -element.

Theorem 1.8: P. Hall [1]

Let G be a locally nilpotent group without π -elements different from the identity. If C is any centralizer in G or any term of the upper central series of G , then $C = C_{\pi}^G$, i.e. C is π -isolated subgroup.

Proof: Suppose first that $C = C(A)$ is the centralizer of the set A of elements of G . Let $y \in C_{\pi}$, then $y^m \in C$, for some π -number m . Hence $(a^{-1}ya)^m = y^m$ for all $a \in A$. But $\{y, a^{-1}ya\}$ is nilpotent. By 1.7 $[y, a] = y^{-1}a^{-1}ya$ is a π -element. Hence $[y, a] = 1$, and $y \in C$, $C = C_{\pi}^G$. The case $C = Z_1 = Z(G)$ is induced by choosing $A = G$. Suppose the result proved for $C = Z_i$, where Z_{i+1}/Z_i is the center of G/Z_i . Then $\bar{G} = G/Z_i$ has no π -element $\neq 1$ and is locally nilpotent. By the case $i = 1$, we have $\bar{C} = \overline{C_{\pi}^G}$ if $\bar{C} = Z(\bar{G}) = Z_{i+1}/Z_i$. Hence $(Z_{i+1})_{\pi}^G = Z_{i+1}$.

As a result of this, the following is true

Corollary 1.9: In a torsion free locally nilpotent group, all centralizers and all terms of the upper central series are π -isolated, for all choices of π .

Now we need the following lemma to prove the last result of this chapter.

Lemma 1.10: P. Hall [1]

Let G be a finitely generated nilpotent group and let $H \leq H_1$ and $K \leq K_1$ be subgroups of G such that $|H_1:H|$ and $|K_1:K|$ are π -numbers. Then $|[H_1, K_1]:[H, K]|$ is a π -number.

As a result of this, if H and K are subgroups of a locally nilpotent group G , then

$$[H_{\pi}^G, K_{\pi}^G] \leq [H, K]_{\pi}^G.$$

Theorem 1.11: P. Hall [1]

Let G be locally nilpotent, H any subgroup of G and N its normalizer in G then:

(i) $H_{\pi}^G \triangleleft N_{\pi}^G$ and if $H = H_{\pi}^G$, then $N = N_{\pi}^G$.

(ii) If G is nilpotent, then

$$H = H_{\pi}^G \text{ if and only if } H = N \cap H_{\pi}^G.$$

(iii) If G is finitely generated then N_{π}^G is the normalizer of H_{π}^G in G .

Proof: (i) $[H_{\pi}^G, N_{\pi}^G] \leq [H, N]_{\pi}^G$ by the preceding lemma and

$[H, N] \leq H$. So $[H_{\pi}^G, N_{\pi}^G] \leq H_{\pi}^G$. Thus $H_{\pi}^G \triangleleft N_{\pi}^G$. Hence if $H = H_{\pi}^G$, we have $H \triangleleft N_{\pi}^G$ and so $N_{\pi}^G = N$.

(ii) Suppose G is nilpotent and $H = H_{\pi}^G \cap N$. Let N_1 be the normalizer of N in G . It will be sufficient to prove that

$N = N_{\pi}^G \cap N_1$; for then, continuing the normalizer series H, N ,

$N_1, \dots, N_r = G$, we obtain $H = H_{\pi}^G \cap N = H_{\pi}^G \cap N_1 = \dots = H_{\pi}^G \cap G = H_{\pi}^G$.

Thus we may take $G = N_1$ so that $N \triangleleft G$. Using a contradiction argument,

we suppose $G = \{N, x\}$ where $x^m \in N$ for some π -number $m > 1$ and

deduce the existence of an element y in $(H_\pi^G \setminus H) \cap N$.

Let $K = \bigcap_{t \in G} H^t$ so that $K \triangleleft G$. Let L/K be the center of N/K since $G = N_\pi^G$, we have $[L, G] \leq [L_\pi^G, N_\pi^G] \leq [L, N]_\pi^G \leq H_\pi^G$ by 1.10, since $[L, N] \leq K \leq H$. If $[L, G] \not\leq H$, we can choose for y any element of $[L, G]$ not in H . But $[L, G] \triangleleft G$ and so if $[L, G] \leq H$, then $[L, G] \leq K$, and therefore $L \triangleleft G$, $L \cap H = K$. But H is not normal in G , and $K < H$. Define $H_0 = H$, $H_{i+1} = [H_i, N]$ for $i \geq 0$. Let j be the first index for which $H_{j+1} \leq K$. Then $H_j \leq L$ and $L \cap H \geq H_j > K$, a contradiction.

(iii) Let G be finitely generated and let x be an element of G such that $x^{-1}H_\pi^G x = H_\pi^G$. We have to prove $x \in N_\pi^G$. Take $G = \{N_\pi^G, x\}$. Then $H_\pi^G < G$ by (i), and the choice of x . By 1.6, $|H_\pi^G : H| = m$ is a π -number. Hence $(H_\pi^G)^m \leq H$. But $(H_\pi^G)^m \triangleleft G$ since $H_\pi^G \triangleleft G$. Let C be the centralizer in G of the finite group $H_\pi^G / (H_\pi^G)^m$. Then $|G : C|$ is finite and since G is nilpotent, $|G : C|$ is a π -number. We have $[H_\pi^G, C] \leq (H_\pi^G)^m \leq H$ so that $C \leq N \leq N$.

Hence $|G : N_\pi^G|$ is a π -number, $G = N_\pi^G$, $x \in N_\pi^G$ as required.

Chapter 2

Fitting Isolated Groups

Introduction 2.1: The work of this chapter has been done by A.H. Rhemtulla and B. Hartley, in order to develop enough isolator theory in polycyclic groups, extend the results of P. Hall which we already introduced in Chapter one and to give a proof to one of Roseblade's recent results on prime ideals in group rings of polycyclic groups. However we gave here a definition to the class of Fitting Isolated groups and a proof to one of their major results which will be used in the following chapter. Also we mention Roseblade's results and will prove it along the lines of Chapter 3.

Following terminology recently proposed by Roseblade [2] we shall call a subgroup H of a group G orbital if H has a finite orbit under the conjugation action of G on the set of subgroups of G , that is, if $|G:N_G(H)| < \infty$. A group G is called orbitally sound if $|H:H_G| < \infty$ for every orbital subgroup H of G , where $H_G = \bigcap_{x \in G} H^x$ is the core of H in G .

By a theorem of Ito and Szep [3, p.45], this is equivalent to saying that $|H^G|H_G| < \infty$, where H^G is the normal closure of H in G .

In Roseblade's recent and fundamental work on prime ideals in group rings of polycyclic groups, a significant role is played by the following fact:

Proposition A: (Roseblade [2])

Every polycyclic group has an orbitally sound subgroup of finite index.

Since every polycyclic group has a torsion-free subgroup of finite index, this is really a theorem about torsion free polycyclic

groups. Now it is rather easy to see that every finitely generated torsion free nilpotent group is orbitally sound by using the theory of isolators developed by P. Hall in [1] which we introduced in chapter one.

Definition 2.2: For any subgroup H of a group G , let $\sqrt{H} = {}^G\sqrt{H}$ denote the set $\{g \in G: g^n \in H \text{ for some } n \geq 1\}$. We say that H has an isolator in G , if $\sqrt{H}$ is a subgroup of G , and that H is isolated in G , if $H = \sqrt{H}$.

When G is finitely generated torsion free nilpotent we know that every subgroup H of G has an isolator $\sqrt{H}$, and that $|\sqrt{H}:H| < \infty$. Also $N_G(\sqrt{H})$ is isolated in G . If H is orbital, then $N_G(H) \leq N_G(\sqrt{H})$, we find that $\sqrt{H} \triangleleft G$. Since $|\sqrt{H}:H| < \infty$, we see that H contains a characteristic subgroup of finite index of $\sqrt{H}$, and hence that $|H:H_G| < \infty$.

Instead of confining ourselves to torsion free polycyclic groups, we shall for the most part work within the somewhat wider class of torsion free soluble groups of finite rank. By the rank of a group we mean here its Mal'cev special or Prüfer rank. Thus, G has finite rank, if there exists an integer $n \geq 0$ such that every finitely generated subgroup of G can be generated by n elements. It is known that every torsion free soluble group of finite rank has a subgroup X of finite index whose derived subgroup X' is nilpotent, see [3,part2,p.138].

Also G has a finite series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$$

in which each factor is either finite or a torsion free abelian group of rank 1. The number of torsion free factors in such a series is an invariant of G which we call the Hirsch number of G and denote by $h(G)$.

Finally, each torsion free soluble group G of finite rank has a unique maximal normal nilpotent subgroup called the Fitting subgroup of G and is denoted by $\mathfrak{F}(G)$.

Definition 2.3: Let G be a torsion free soluble group of finite rank. We say that G is FI (Fitting Isolated), if, whenever $H = K|L$ ($L < K \leq G$) is a torsion free section of G , then $H|\mathfrak{F}(H)$ is torsion free abelian.

The class of FI-groups is clearly closed under taking subgroups and torsion free homomorphic images.

Theorem 2.4: [A.H. Rhemtulla and B. Hartley]

If G is any torsion free soluble group of finite rank, then G contains an FI-subgroup of finite index.

Proof: We may assume that $G|\mathfrak{F}(G)$ is abelian. Let

$1 = Z_0 < Z_1 < \dots < Z_c = \mathfrak{F}(G)$ be the upper central series of $\mathfrak{F}(G) = F$, and $C_i = C_G(Z_i/Z_{i-1})$, $1 \leq i \leq c$. Then $F = \bigcap_{i=1}^c C_i$, and each of the

groups $G|C_i$ can be thought of as an abelian group of matrices over $\mathbb{Q}$. The torsion subgroup of each G/C_i is finite [4.9.33], and it follows that the torsion subgroup T/F of G/F is finite. Hence G/F splits over T/F , and there exists a subgroup G_1 of finite index in G , such that G_1/F is torsion free. Since $F = \mathfrak{F}(G_1)$, we may even assume that $G/\mathfrak{F}(G)$ is torsion free abelian.

The proof now falls into two parts:

- (1) There exists a subgroup G_0 of G such that $G_0 \geq F(G)$, $|G:G_0| < \infty$, and $H/F(H)$ is torsion free, for every $H \leq G_0$.
- (2) G_0 is F.I.

We prove (1) by induction on $h(G)$. If $h(G) = 0$ there is nothing to do, so assume $h(G) > 0$. Let B be an abelian normal subgroup of minimal rank of G contained in the centre $Z(F(G))$ of $F(G)$, and let $A = {}^G\sqrt{B}$. Then $A \leq F(G)$ since $G/F(G)$ is torsion free, and hence, by the theory of isolators in nilpotent groups, [1], A is a subgroup of $Z(F(G))$. By induction, there is a subgroup G_1/A of G/A , containing $F(G/A)$ and hence $F(G)A/A$, such that $|G:G_1| < \infty$ and G_1/A satisfies (1).

Let $C = C_G(A)$. Then $C \geq F(G)$, we have seen above that the torsion subgroup of G/C is finite, so that G/C contains a torsion free subgroup G_2/C of finite index. Let $G_0 = G_1 \cap G_2$. Then $|G:G_0| < \infty$ and $G_0 \geq F(G)$.

Now let $H \leq G_0$. If $H \cap A = 1$, then $H \cong HA/A$, and the fact that $H/F(H)$ is torsion free follows from the properties of G_1/A . Suppose that $H \cap A \neq 1$, and let h be an element of H such that $h^n \in F(H)$, for some $n \geq 1$. Since $H \cap A \leq F(H)$, we have $[H \cap A, h^n, \dots, h^n] = 1$, and hence $C_{H \cap A}(h^n) \neq 1$. Hence $C_A(h^n)$ is a non-trivial isolated subgroup of A , and is normal in G since G/C is abelian. Therefore $h^n \in C$. But $G_0/G_0 \cap C$ is torsion free. Hence $h \in C$. If $F/A \cap H = F(H/A \cap H)$, we also know by induction that $h \in F$. Hence $h \in C_F(A \cap H)$, a nilpotent normal subgroup of H . Therefore $h \in F(H)$, as required.

Next we will prove (2). We have to show that if G is a torsion

free soluble group of finite rank and

(*) $H/\underline{F}(H)$ is torsion free abelian for every $H \leq G$,

then G/N has the property (*) whenever $N \triangleleft G$ and G/N is torsion free.

Suppose this is, false, and let r be the smallest integer for which there exists a counterexample G with $h(G) = r$. Among all pairs (G, N) which furnish a counterexample with $h(G) = r$, choose one with $h(N)$ minimal. Then G/N contains a subgroup H/N such that $(H/N)/\underline{F}(H/N)$ is not torsion free. We may clearly assume that $H = G$.

Let $F/N = \underline{F}(G/N)$, and choose an element $t \in G/F$ such that $t^m \in F$ for some $m > 0$. Let $G_1 = \underline{F}(G)\langle t \rangle$, $N_1 = N \cap G_1$, $F_1 = F \cap G_1$. Then $G_1/N_1 \cong G_1N/N$, under this isomorphism, F_1/N_1 corresponds to $(F \cap G_1)N/N = (F/N) \cap (G_1N/N) = \underline{F}(G_1N/N)$, as $G_1N/N \triangleleft G/N$. We have $t \in G_1$, $t \notin F_1$, $t^m \in F_1$. Thus (G_1, N_1) is a counterexample, so we may assume that $G = G_1$, that is $G = \underline{F}(G)\langle t \rangle$

Next we notice that, if $Z = \underline{Z}(\underline{F}(G))$, then

$$C_Z(t) = 1 \quad (1)$$

clearly $C_Z(t) = \underline{Z}(G)$. Let Y denote this subgroup, which is isolated in G since Z is. By the minimality of $h(N)$, there is no normal subgroup M of G such that $1 < M < N$ and G/M is torsion free. Clearly $N \cap \underline{F}(G) \neq 1$, and so $N \cap Z \neq 1$. Since Z is isolated in $\underline{F}(G)$, [1], and $G/\underline{F}(G)$ is torsion-free, Z is isolated in G , so $N \leq Z$. Hence $YN \leq Z$, and the isolator ${}^G\sqrt{YN}$ of YN in G is an abelian normal subgroup of G . Also $\sqrt{YN}/N = V$ is a torsion free abelian normal subgroup of G/N , and $V/C_V(G)$ is periodic, since $C_V(G) \geq YN/N$. Hence $[V, G] = 1$, that is $\sqrt{YN}/N \leq Z(G/N)$.

Now trivial arguments show that G/Y has the property (*). If $Y \neq 1$, then the minimality of $h(G)$ shows that $G/\sqrt{YN}$ has the property (*). But $\sqrt{YN}/N \leq F(G/N) = F/N$, and $F(G/\sqrt{YN}) = F/\sqrt{YN}$ as $\sqrt{YN}/N \leq Z(G/N)$. Hence G/F is torsion free, a contradiction. This proves (1).

Clearly $Z/N \leq F/N$, and since $t^m \in F$, we have

$$[Z, t^m, \dots, t^m] \leq N.$$

In particular, if $Z > N$, then $C_{Z/N}(t^m) = K/N \neq 1$. Now commutation with t^m induces endomorphism ξ of K whose image lies in N and so has smaller rank than K . Hence $\ker \xi = C_K(t^m) \neq 1$. Let $L = C_K(t^m)$. The $t^m \in \underline{F}(L\langle t \rangle)$, and by (*), $t \in \underline{F}(L\langle t \rangle)$, that is $L\langle t \rangle$ is nilpotent. Therefore $[L, t, \dots, t] = 1$, and $C_L(t) \neq 1$. This contradicts (1). We deduce that $Z = N$.

It clearly follows that $\underline{F}'(G) \neq 1$. Let $U/\underline{F}'(G)$ be the torsion subgroup of $\underline{F}(G)/\underline{F}'(G)$. Then $N \leq U$, as $N \cap \underline{F}'(G) \neq 1$. Since $\underline{F}(G) \leq G$, we deduce that

$$[\underline{F}(G), t^m, \dots, t^m] \leq U.$$

Let C be the nilpotency class of $\underline{F}(G)$. Then

$$1 \neq \gamma_C(\underline{F}(G)) \leq N,$$

where $\{\gamma_i(X)\}$ is the lower central series of a group X .

If $x_1, \dots, x_c \in \underline{F}(G)$, then since $\gamma_C(\underline{F}(G))$ is torsion free, the value of $[x_1, \dots, x_c]$ only depends on the value of $x_1, \dots, x_c$ modulo U . We obtain a well-defined $\langle t \rangle$ -module epimorphism of $\underline{F}(G)/U \otimes \dots \otimes \underline{F}(G)/U$ (with c factors) onto $\gamma_C(\underline{F}(G))$, namely

$$x_1 U \otimes \dots \otimes x_c U \rightarrow [x_1, \dots, x_c] \quad .$$

(cf. [3,part1,p.55]). The tensor product is to be viewed as a $\langle t \rangle$ -module via the diagonal action, the action on the individual factors being by conjugation. Since $\mathbb{F}(G)/U$ has a finite series with t^m -trivial factors, so does the tensor product, and hence also $\gamma_c(\mathbb{F}(G))$ (cf. [3],part 1, p.56). Hence $C_N(t^m) \neq 1$. Arguing as in the previous paragraph, we deduce that $C_N(t) \neq 1$, and obtain a contradiction to (1). This concludes the proof.

Chapter 3

Isolators in Polycyclic Groups

Introduction: The work of this Chapter is the first step in an attempt to extend P. Hall results in [1]. If G is not nilpotent group, we show in 3.2.1 that, there is no reason to expect ${}^G\sqrt{H}$ to be always a subgroup. In 3.2.2, we show that if G is polycyclic Fitting Isolated group then G has a subgroup G_0 of finite index, such that ${}^{G_0}\sqrt{H}$ is a subgroup for every $H \leq G_0$ and that $|{}^{G_0}\sqrt{H}:H|$ is finite. Also throughout the lines of this Chapter we give a proof to proposition (A) of Chapter 2.

If G is polycyclic group, Mal'cev in an earlier theorem, has proved that G can be embedded in $GL(n, \mathbb{Z})$, the general linear group of degree n over the integers. And since every linear group is a CZ-group, we need here to study those properties of CZ-groups that seem relevant to the theory of isolators.

I have therefore split this chapter into CZ-groups in section 1 and the isolator theory of polycyclic groups in section 2.

Section 1

CZ-groups and the Zariski topology

Let U be the space of n -row vectors over the field F and $R = F[X_1, \dots, X_n]$, the polynomial ring over F in n indeterminants. A subset A of U is said to be closed in U if there exists a subset S of R such that A is the set of zeros of S , that is, if

$$A = \{(a_1, \dots, a_n) \in U: f(a_1, \dots, a_n) = 0 \text{ for all } f \text{ in } S\}.$$

If S is any subset of R let $V(S)$ denote the set of zeros of S (in U). Note that $V(S) = V(\text{ideal generated by } S)$

$$\bigcap_{\alpha} V(S_{\alpha}) = V\left(\bigcup_{\alpha} S_{\alpha}\right).$$

Suppose that $A = V(I)$ and $B = V(J)$ where I and J are ideals. We claim that $A \cup B = V(IJ)$. For $IJ \subseteq I$, so $A = V(I) \subseteq V(IJ)$, and similarly for B . Conversely let $x \in V(IJ) \setminus A$. Since $A = V(I)$ there exists an element f of I such that $f(x) \neq 0$. If $g \in J$, then $fg \in IJ$, so $0 = fg(x) = f(x)g(x)$. Hence $g(x) = 0$ for all g in J and so $x \in V(J) = B$.

Thus the closed subsets of U define a topology on U , called the Zariski topology. Note that $U = V(\{0\})$, $\phi = V(\{1\})$. If $(a_1, \dots, a_n) \in U$ then $\{(a_1, \dots, a_n)\} = V(\{x_1 - a_1, \dots, x_n - a_n\})$. Hence every one element subset of U is closed (i.e. the topology is a T_1 -topology). If $A \subseteq B$ are closed sets there exists ideals I and J of R such that $A = V(I)$, $B = V(J)$ and $J \subseteq I$. For if $A = V(I')$ and $B = V(J)$, put $I = I' + J$. Thus the Hilbert basis theorem implies that the Zariski topology satisfies the descending chain condition on closed sets, or equivalently the ascending chain condition on open sets.

Definition 3.1.1: A Z-space is a topological space in which every one element set is closed (i.e. is a T_1 -space) and which satisfies the descending chain condition on closed sets.

Every subspace of a Z-space is a Z-space.

Definition 3.1.2: A CZ-group is a Z-space G whose underlying set carries a group structure such that for every a in G the four

mappings given by

$$\begin{aligned} x \rightarrow ax \quad , \quad x \rightarrow x^{-1} \\ x \rightarrow xz \quad \quad x \rightarrow x^{-1}ax \quad , \quad x \in G \end{aligned}$$

are continuous.

Every linear group is a CZ-group, for if V is an arbitrarily finite dimensional vector space over F , then V can be endowed with a Zariski topology after a suitable choice of basis.

Let G be a subgroup of $GL(n, F) \subseteq F_n$. F_n is a vector space over F of dimension n^2 and so carries the Zariski topology. The four mappings above are continuous [4, 5.1, page 72].

Since the topology induced on a given linear group is unaffected by ground field extension [4, page 73], we can unambiguously speak of its closed subsets without specifying the ground field.

Theorem 3.1.3: [4, page 75]

If G is CZ-group then the normalizer of a closed subset of G is closed and the centralizer of any subset of G is closed.

Proof: Let S be any closed subset of G and a any element of S . Denote by $S(a)$ the inverse image of S in G under the continuous mapping given by $x \rightarrow x^{-1}ax$. Then $N_1 = \bigcap_{a \in S} S(a) = \{x \in G: x^{-1}Sx \subseteq S\}$ is closed in G .

If $(a)S$ denotes the inverse image of S in G under the continuous mapping given by $x \rightarrow xax^{-1} = (x^{-1})^{-1}ax^{-1}$, then

$N_2 = \bigcap_{a \in S} (a)S = \{x \in G: xSx^{-1} \subseteq S\}$ is closed. Therefore $N_G(S) = N_1 \cap N_2$

is closed in G .

Let T be any subset of G .

$$C_G(T) = \bigcap_{t \in T} C_G(t) = \bigcap_{t \in T} N_G(t)$$

and $N_G(t)$ is closed by the above. Consequently $C_G(T)$ is closed in G .

Theorem 3.1.4: [4, page 78]

In a CZ-group G the closure of a subgroup is a subgroup and the closure of a normal subgroup is a normal subgroup.

Proof: Let H be a subgroup of G and $\bar{H}$ its closure in G . Now $H = H^{-1}$. Since the mapping given by $x \rightarrow x^{-1}$ is a homeomorphism of G , $(\bar{H})^{-1} = \bar{H}$. Let $h \in H$. The inverse image of $\bar{H}$ in G under the mapping given by $x \rightarrow hx$ is closed and contains H . Therefore it contains $\bar{H}$; that is, $h\bar{H} \subseteq \bar{H}$ for all h in H . If $k \in \bar{H}$, the inverse image of $\bar{H}$ under the mapping given by $x \rightarrow xk$ contains H and so $\bar{H}$. Hence $\bar{H} \cdot \bar{H} \subseteq \bar{H}$ is a subgroup of G . Suppose that H is a normal subgroup of G . Since the mapping given by $x \rightarrow a^{-1}xa$ is continuous, $a\bar{H}a^{-1}$ is a closed subset of G containing H . Therefore $\bar{H} \subseteq a\bar{H}a^{-1}$ for all a in G ; that is $\bar{H}$ is normal in G .

Theorem 3.1.5: [p, page 78]

Let A, B and C be subgroups of the CZ-group G and $\bar{A}, \bar{B}$ and $\bar{C}$ their closures in G . If $[A, C] \leq B$ then $[\bar{A}, \bar{C}] \leq \bar{B}$.

Proof: Let $c \in C$. The mapping given by $x \rightarrow [a, c]$ is continuous and the inverse image of $\bar{B}$ in G contains A . Therefore $[\bar{A}, c] \leq \bar{B}$ for all c in C .

If $a \in \bar{A}$ the inverse image of $\bar{B}$ under the mapping given by $x \rightarrow [a, x]$ contains C and hence $\bar{C}$. Thus $[\bar{A}, \bar{C}] \leq \bar{B}$.

Theorem 3.1.6: [Wehrfritz]

R is finitely generated integral domain of characteristic $p \geq 0$ and $G = GL(n, R)$, then if $p = 0$, for all but finitely many primes q , then G contains a subgroup T_q of finite index in G such that for all closed subgroups H of G

$$(N_G(H) \cap T_q)H/H \text{ is residually finite } q\text{-group.}$$

For the proof see [5]. As a matter of fact we can take

$T_q = G \cap (I + qM(n, \mathbb{Z}))$, q any prime, where I is the identity matrix of rank n , and $M(n, \mathbb{Z})$ is the set of n by n matrices over the integers $\mathbb{Z}$.

$$\text{Then } T_q = \{g \in G: g \equiv 1 \pmod{q}\}.$$

Lemma 3.1.7 If K is residually finite p -group and residually finite q -group then K is torsion free.

Proof: Let $x \in K$, $x \neq e$, $x^n = e$, $n \geq 2$. Let $n = p^i q^j m$, where $(p, m) = (q, m) = 1$. and $q^j m \neq 1$ or $p^i m \neq 1$. Assume without loss of generality that $q^j m \neq 1$. $\therefore x^{p^i}$ has order $q^j m$, replace x by $x^{p^i} = y$ so that $y \neq 1$ and order y is $q^j m$, where $(p, q^j m) = 1$. Now, $y \in K$, which is residually finite p -group, implies $\exists H < K$ such that $y \notin H$ and K/H is a finite p -group. So $H y$ has order p^α (i.e. $y^{p^\alpha} \in H$, for some integer α) and $(p^\alpha, q^j m) = 1$ hence $\exists$ integers r and s such that $rp^\alpha + sq^j m = 1$ i.e. $y = y^{rp^\alpha} \cdot y^{sq^j m} = y^{rp^\alpha} \in H$ contradiction.

So if we let $G_0 = G \cap (I + 2M(n, \mathbb{Z})) \cap (I + 3M(n, \mathbb{Z}))$ then by 3.1.6, $N(H)/H$ is residually finite 2-group and residually finite 3-group, i.e. $N(H)/H$ is torsion free for all closed subgroups H of G_0 .

Corollary 3.1.8: If H is a closed subgroup of G_0 , then H is isolated.

Proof: Suppose H is not isolated in G_0 , let $x^r \in H$ for some integer $r > 1$ and $x \notin H$. Let $J =$ the closure of $\langle x^r \rangle$ (the Zariski closure in G_0). Then $J \leq H$, and since x normalizes $\langle x^r \rangle$ then $x \in N(J)$, and we have $x^r \in J$. But from the remark made above since J is closed then $N(J)/J$ is torsion free. Then $x \in J$ i.e. $x \in H$ contradiction.

Definition 3.1.9: Let R be a commutative ring and M a finitely generated R -module. An element g of $\text{Aut}_R(M)$ is called unipotent if $(g-1)$ is nilpotent, i.e. if some power of $(g-1)$ is the zero map. A subgroup G of $\text{Aut}_R(M)$ is called unipotent if every element of G is unipotent.

Theorem 3.1.10: [Wehrfritz]

If H is a subgroup of $U(G)$, the set of unipotent elements of G , where $G = \text{GL}(n, Q)$. Then the Zariski closure $\bar{H}$ of H in G is equal to the isolator of H in $U(G)$.

For the proof see [6].

2. Isolators in Polycyclic Groups

There is no reason to expect ${}^G\sqrt{H}$ to be always a subgroup, for

Example 3.2.1: Let $G = \langle a, b: a^b = a^{-1} \rangle$. Thus $[b^2, a] = 1$, $(ab)^2 = b^2$. Let $H = \langle b^2 \rangle$, then ${}^G\sqrt{H} \geq \{ab, b\}$. But $G = \langle ab, b \rangle$ and $G \neq {}^G\sqrt{H}$, for $a \notin {}^G\sqrt{H}$.

If G is locally nilpotent group P. Hall, as we have seen in chapter 1, has proved the following:

- i) ${}^G\sqrt{H}$ is always a subgroup
- ii) If $H = {}^G\sqrt{H}$, then $N_G(H) = {}^G\sqrt{N_G(H)}$ i.e. if H is isolated subgroup, so is $N_G(H)$.
- iii) If G is finitely generated, and therefore nilpotent, then $|{}^G\sqrt{H}:H| < \infty$.
- iv) If G is finitely generated, and therefore nilpotent, then ${}^G\sqrt{N_G(H)} = N_G({}^G\sqrt{H})$.

Theorem 3.2.2: [Rhemtulla and Wehrfritz].

Every polycyclic group G has a subgroup G_0 of finite index in G such that

- i) $H \leq G_0$, then ${}^{G_0}\sqrt{H}$ is a subgroup.
- ii) If $H = {}^{G_0}\sqrt{H}$, then $N_{G_0}(H) = {}^{G_0}\sqrt{N_{G_0}(H)}$
- iii) $|{}^{G_0}\sqrt{H}:H| < \infty$.
- iv) ${}^{G_0}\sqrt{N_{G_0}(H)} = N_{G_0}({}^{G_0}\sqrt{H})$.

Corollary 3.2.3: Roseblade's result in Chapter 2, follows from 3.2.2.

Proof: If G_0 is given as in 3.2.2 then G_0 is orbitally sound, for, let H be a subgroup of G_0 and $N = N_{G_0}(H)$ with $|G_0:N| < \infty$, then $G_0 = {}^{G_0}\sqrt{N}$. But from (ii), we get

$$G_0 = N_{G_0}({}^{G_0}\sqrt{H}) \quad \text{i.e.} \quad {}^{G_0}\sqrt{H} \triangleleft G_0,$$

call $I = {}^{G_0}\sqrt{H}$, hence from (iii) we have $I^m \leq H$, $m \neq 0$. Then if $a \in I$ then $a^g \in I$ (since $I < G$), $g \in G_0$. Hence $(a^g)^m = (a^m)^g \in H$ for every $g \in G_0$. Hence $a^m \in H^G$, for every $g \in G_0$ i.e.

$a^m \in \bigcap_{g \in G_0} H^g = \text{core}_{G_0} H = H_{G_0}$ which implies I/H_{G_0} is periodic, i.e.

H/H_{G_0} is periodic and since G_0 is polycyclic, then H/H_{G_0} is finite, which gives the result.

Proof of 3.2.4: We shall show the proof on steps.

Step 1. Mal'cev has proved that every polycyclic group G can be embedded in $GL(n, \mathbb{Z})$, the general linear group of degree n over the integers, also G is nilpotent by abelian by finite, and the nilpotent by abelian part is triangularizable over the algebraic closure of $\mathbb{Q}$, see [4, Theorem 3.6, page 45].

Thus we pass from G to this triangularizable subgroup and we call it G again and let $U(G)$ be the set of unipotent matrices of G . By the definition of $U(G)$ it is easy to see that it is a normal subgroup of G and by [2, Theorem 13.6, page 189], $U(G)$ is nilpotent. And by a theorem of Gruenberg [7] we may assume that $G' \leq U(G)$.

By 2.4, we may also assume that G is Fitting isolated.

Step 2. If $G_0 = G \cap (I + 2M(n, \mathbb{Z})) \cap (I + 3M(n, \mathbb{Z}))$ then by 3.1.6, G_0 is of finite index in G and G_0 satisfies the following properties:

i) $N_{G_0}(H)/H$ is torsion free for every closed subgroup H of G , by the remark made after Lemma 3.1.

ii) $U(G_0)$ is closed subgroup of G_0 , by 3.5, with $U(G_0) \triangleleft G_0$, $U(G_0)$ nilpotent and $G'_0 \leq U(G_0)$, from Step 1.

We shall try to show that G_0 satisfies i, ii, iii and iv mentioned in 3.2.2.

Step 3. $U(G_0)$ is isolated by Corollary 3.1.8. Let $U(H) = H \cap U(G_0)$, where $H \leq G_0$ and suppose H is isolated subgroup. Then $U(H)$ is also isolated, since both H and $U(G_0)$ are so. Then by 3.1.10,

$U(H)$ is closed, and since $G'_0 \leq U(G_0)$, then $[N(H), H] \leq H \cap U(G_0) = U(H)$ (by $N(H)$ we mean here $N_{G_0}(H)$). Applying 3.1.5, we get $[\overline{N(H)}, \overline{H}] \leq \overline{U(H)}$.

But $U(H) = \overline{U(H)}$, then $[\overline{N(H)}, H] \leq U(H) \leq H$. i.e. $[\overline{N(H)}, H] \leq H$, then $N(H) = \overline{N(H)}$ by the definition of the normalizer, i.e. $N(H)$ is closed. Hence $N(H)$ is isolated by Corollary 3.18.

Hence we conclude that if H is isolated subgroup of G_0 so is $N(H)$, which proves (ii).

(*) We can also see from the preceding argument, that if H is any subgroup of G_0 and $U(H)$ is isolated in $U(G_0)$ then $[\overline{N(H)}, H] \leq U(H) \leq H$.

Step 4. Now take $H \leq G_0$, $U(H) = H \cap U(G_0)$.

$$\begin{aligned} G_0 \sqrt{U(H)} &= U(G_0) \sqrt{U(H)}, \text{ since } G_0/U(G_0) \text{ is torsion-free} \\ &= \text{The closure of } U(H) \text{ in } G_0, \text{ by 3.1.10} \\ &= \overline{U(H)} \end{aligned}$$

since $U(G_0)$ is nilpotent, then $\overline{U(H)}$ is a subgroup and $|\overline{U(H)} : U(H)| < \infty$, by 1.5. Since $[N(H), H] \leq U(H)$, for every subgroup H of G_0 then

$[N(U(H)), U(H)] \leq U(H)$. But $N(H) \leq N(U(H))$, then

$[N(H), U(H)] \leq U(H)$. i.e. $U(H) \triangleleft N(H)$. Hence $\overline{U(H)} \triangleleft \overline{N(H)}$,

for if $x \in \overline{U(H)}$, $g \in N(H)$, then $x^n \in U(H)$, $n \neq 0$ and

$g^{-1}x^ng = (g^{-1}xg)^n \in U(H)$, since $U(H) \triangleleft N(H)$ then $g^{-1}xg \in \overline{U(H)}$. i.e. $[N(H), \overline{U(H)}] \leq \overline{U(H)}$. Applying 2.1.5, then $[\overline{N(H)}, \overline{U(H)}] \leq \overline{U(H)}$. Now let $H_1 = H \cdot \overline{U(H)}$ since $\frac{H \cdot \overline{U(H)}}{H} \cong \frac{\overline{U(H)}}{H \cap \overline{U(H)}}$ and $|\overline{U(H)} : U(H)| < \infty$, then $|H_1 : H| < \infty$.

Since $U(H_1) = H_1 \cap U(G_0) = H \cdot \overline{U(H)} \cap U(G_0) = \overline{U(H)} \cdot (H \cap U(G_0)) = \overline{U(H)} \cdot U(H) = \overline{U(H)}$ then $U(H_1)$ is closed, i.e. it is isolated. Then by (*) $[\overline{N(H_1)}, H_1] \leq H_1$ i.e. $[\overline{N(H_1)}, H_1] \leq H_1 \cap U(G_0) = U(H_1)$ and since $H_1 \leq N(H_1) \Rightarrow \overline{H_1} \leq \overline{N(H_1)}$, then $[\overline{H_1}, H_1] \leq U(H_1)$. Hence $H_1/U(H_1) \leq Z(\overline{H_1}/U(H_1)) =$ the center of the group $\overline{H_1}/U(H_1)$. Now since $U(H_1)$ is isolated, then $\overline{H_1}/U(H_1)$ is torsion free, then it is Fitting isolated group. And since $Z(\overline{H_1}/U(H_1)) \leq$ Fitting subgroup of $(\overline{H_1}/U(H_1))$, which is isolated in $\overline{H_1}/U(H_1)$, then $Z(\overline{H_1}/U(H_1))$ is isolated. Hence it is closed subgroup and since $H_1/U(H_1) \leq Z(\overline{H_1}/U(H_1))$ then $\overline{H_1}/U(H_1) \leq Z(\overline{H_1}/U(H_1)) \leq \overline{H_1}/U(H_1)$ i.e. $\overline{H_1}/U(H_1)$ is abelian, and since $U(H_1) \leq H_1$ then $\overline{H_1}/H_1$ is abelian which is periodic and finitely generated, then it is finite i.e. $|\overline{H_1} : H_1| < \infty$. But $|H_1 : H| < \infty$, then $|\overline{H_1} : H| < \infty$ since $\overline{H} \leq \overline{H_1}$, then $|\overline{H} : H| < \infty$, and we know before, from 3.1.4, that $\overline{H}$ is a subgroup. And since $\overline{H}$ is closed then, by Corollary 3.1.8, it is isolated in G_0 . i.e. $\overline{H} = G_0 \sqrt{\overline{H}}$ and since $G_0 \sqrt{\overline{H}} = G_0 \sqrt{H}$ then $\overline{H} = G_0 \sqrt{H}$, i.e. $G_0 \sqrt{H}$ is a subgroup of G_0 and $|G_0 \sqrt{H} : H| < \infty$, which proves (i) and (iii).

To see (iv), since $[N(\overline{H}), \overline{H}] \leq \overline{H}$, applying 3.1.5 then $[\overline{N(\overline{H})}, \overline{H}] \leq \overline{H}$, i.e. $\overline{N(\overline{H})} = N(\overline{H})$, hence $N(\overline{H})$ is isolated. And since $N(H) \leq N(\overline{H})$, then $\overline{N(H)} \leq \overline{N(\overline{H})} = N(\overline{H})$ i.e. $\overline{N(H)} \leq N(\overline{H})$.

Now we want to prove $N(\overline{H}) \leq \overline{N(H)}$. Let $x \in G_0$ such that $x^{-1}\overline{H}x = \overline{H}$, we have to prove $x \in \overline{N(H)}$. Take $K = \{\overline{N(H)}, x\}$. Then $\overline{H} \triangleleft K$ since $[N(H), H] \leq H$ and applying Theorem 3.1.5, we get $[\overline{N(H)}, \overline{H}] \leq \overline{H}$, i.e. $\overline{H} \triangleleft \overline{N(H)}$ and by the choice of x , then $\overline{H} \triangleleft K$

By (iii), $|\bar{H}:H| = m$, where m is a finite number. Hence $\bar{H}^m \leq H$. But $\bar{H}^m \triangleleft K$ since $\bar{H} \triangleleft K$. Let C be the centralizer in K of the finite group $\bar{H}/\bar{H}^m$. Then $|K:C|$ is finite. We have $[\bar{H}, C] \leq \bar{H}^m \leq H$ so that $C \leq N(H) \leq \overline{N(H)}$. Hence $|K:\overline{N(H)}|$ is finite i.e. $K = \overline{N(H)}$ and $x \in \overline{N(H)}$ as required.

Now the following Corollary will, in some extent, generalize Theorem 3.1.10.

Corollary 3.2.5: Every polycyclic group G has a subgroup G_0 of finite index, such that, if H is a subgroup of G_0 , then the Zariski closure $\bar{H}$ of H in G_0 is equal to the isolator of H in G_0 .

Proof: If $G_0 = G \cap (I+2M(n, \mathbb{Z})) \cap (I+3M(n, \mathbb{Z}))$ then by 3.1.6, G_0 is of finite index in G . And if $H \leq G_0$, then 3.1.4, the Zariski closure $\bar{H}$ of H in G_0 is a subgroup and as we have seen throughout the proof of 3.2.4, $|\bar{H}:H|$ is finite. And since $\bar{H}$ is closed then by Corollary 3.1.8, it is isolated in G_0 , i.e. $\bar{H} = {}^{G_0}\sqrt{\bar{H}}$. But since ${}^{G_0}\sqrt{\bar{H}} = {}^{G_0}\sqrt{H}$, then $\bar{H} = {}^{G_0}\sqrt{H}$.

Chapter 4

Π -Isolators and Polycyclic Groups

In this Chapter we present the main result of this thesis.

We define a class χ of groups as follows:

Definition 4.1: We say a group G is in χ if G is soluble of finite rank and there exists a subgroup G_0 of finite index in G and a finite set of primes π_1 such that whenever $\pi_1 \subseteq \pi$ (a set of primes) and $K \leq G_0$ then the set $\{g \in G: g^m \in K, \text{ for some } \pi\text{-number } m\}$ is a subgroup (called the π -isolator of K in G_0 and denoted by $K_{\pi}^{G_0}$), and $|K_{\pi}^{G_0}:K|$ is a π -number.

Now the following theorem is an immediate consequence of the definition.

Theorem 4.2: i) If $G \in \chi$ and $H \leq G$, then $H \in \chi$.

ii) If $G \in \chi$ and $H \triangleleft G$, then $G/H \in \chi$.

iii) If $H \in \chi$ and $K \in \chi$, then $H \times K \in \chi$.

iv) If G is finitely generated nilpotent by finite group then $G \in \chi$, see 1.5.

One could conceivably try to extend P. Hall result in 1.5 to some other class of groups by investigating the class χ . The following example shows that some metabelian polycyclic groups already fail to be in χ .

Theorem 4.3: If $G = \langle x, y, t: xy = yx, x^t = y, y^t = xy \rangle$ then $G \notin \chi$.

Proof: The subgroup $\langle x, y \rangle$ of G represents the Fitting subgroup, $F(G)$, of G . And $G > F(G) > 1$ is a normal series of G with $G/F(G) \cong \langle t \rangle$ the infinite cyclic group and $F(G) = \langle x, y \rangle$ is torsion

free abelian of rank 2. That is to say G is a metabelian polycyclic group.

Now suppose $G \in \chi$. We may assume that π_1 has largest prime $p > 5$, by enlarging π_1 if necessary, we may also assume that G_0 contains $\mathbb{F}(G)$. i.e. $G_0 = \langle x, y, t^m \rangle$, for some $m > 0$.

Let $N = \langle x, y^q, t^{mp} \rangle$ be a subgroup of G_0 , where q is a prime, which will be chosen later.

Question (*): If N is given as above, is there a prime q greater than p , such that N is normalized by t^{mp} but not by t^m ?

If a positive answer were given to (*) then we are finished. For, let $t^m = \tau$ and suppose $\exists q > p$ such that $N = \langle x, y^q, \tau^p \rangle$ is normalized by τ^p but not by τ . Let $L = N \cap \mathbb{F}(G)$, $M = L^{G_0}$ (the normal closure of L in G_0), then it is easy to see that $M = \mathbb{F}(G)$ and $G_0 = \langle L, \tau \rangle$. Since N is not normal in G_0 , then N is a proper subgroup of $\mathbb{F}N$. So if $q \notin \pi$, where π is a set of primes containing π_1 , then $N_\pi^{G_0}$ is not a subgroup. For, if $N_\pi^{G_0}$ is a subgroup, so it must be all of G_0 , and $|G_0 : N|$ is a finite π -number. But then $y^m \in N$, where m is π -number, and so $y^q \in N$ implies $y \in N$, a contradiction.

And if we throw q inside π_1 then by repeating the argument we will be able to find an increasing sequence of primes

$q = q_1 < q_2 < \dots < q_i < \dots$ which will prevent us from keeping π_1 finite if we want (i) $N_\pi^{G_0}$ to be a subgroup, where
 (i) $N = \langle x, y^{q_{i+1}}, \tau^{q_i} \rangle$, $i = 0, 1, 2, \dots, q_0 = p$.

Now by studying the action of the operator t on $\mathbb{F}(G)$ we can see that $t^2 = t + 1$ and $t^n = F_{n-1} + F_n t$, where F_n are the Fibonacci numbers defined as follows:

$$F_0 = 0$$

$$F_1 = F_2 = 1$$

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0.$$

Now (*) is equivalent to say:

Question (**): Is there a prime $q > p$ such that $q \mid F_{mp}$ but $q \nmid F_m$?

For, if t^{mp} normalizes N , then $x^{t^{mp}} \in N$, i.e. $x^{F_{mp}-1+F_{mp}t} \in N$, then $x^{F_{mp}t} \in N$, i.e. $y^{F_{mp}} \in N$ i.e. $q \mid F_{mp}$; also t^m does not normalize N means $q \nmid F_m$.

Hence in order to give an answer to (**), we need to study the ring of integers $\mathbb{Z}[\theta]$, where $\theta^2 = \theta + 1$. Let $K = \mathbb{Q}(\sqrt{5})$, i.e. K is a quadratic field. $\theta = \frac{1 \pm \sqrt{5}}{2}$, where the minimal polynomial of θ over K is $x^2 - x - 1$. $R = \mathbb{Z}[\theta]$ is a Dedekind ring, for $\{1, \theta\}$ is a basis for R over $\mathbb{Z}$, and the discriminant of R over $\mathbb{Z}$ is equal to the discriminant of the basis $\{1, \theta\}$ which is given by:

$$\det \begin{bmatrix} \text{Tr}(1) & \text{Tr}(\theta) \\ \text{Tr}(\theta) & \text{Tr}(\theta^2) \end{bmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5.$$

Since $\text{disc.}(R/\mathbb{Z}) = 5$, is square free positive integer, then R is Dedekind, see [8]. And since 5 is the only prime dividing $\text{disc.}(R/\mathbb{Z})$, then 5 is the only prime which ramifies in K .

Let q be a rational prime number since R is a Dedekind ring, then qR , as an ideal of R can be factorized as, $qR = Q_1^{e_1} \dots Q_g^{e_g}$, a product of a distinct primes of R with powers $e_i \geq 1$ in $\mathbb{Z}$. The power $e_i = e(Q_i/q\mathbb{Z})$ is called the ramification index of Q_i over $q\mathbb{Z}$.

Since $Q_i \cap \mathbb{Z} = q\mathbb{Z}$ the inclusion $\mathbb{Z} \rightarrow R$ induces a monomorphism

$\mathbb{Z}/q\mathbb{Z} \rightarrow R/Q_i$ of fields. Since R is finitely generated as $\mathbb{Z}$ -module the dimension: $f_i = \dim_{\mathbb{Z}/q\mathbb{Z}} R/Q_i$ is an integer ≥ 1 . The dimension $f_i = f(Q_i/q\mathbb{Z})$ is called the degree of inertia of Q_i over $q\mathbb{Z}$.

We denote by $N_K(I)$ to the cardinality of R/I where I is a non-zero ideal of R .

The integers e_i and f_i satisfy the following equation

$$2 = \sum_{i=1}^g e_i f_i, \text{ see [8], and the later has only three solutions in}$$

positive integers.

1) $g = 1, e_1 = 2, f_1 = 1$. Then $qR = Q^2$, $N_K(Q) = q$ and we say q ramifies in K .

2) $g = 1, e_1 = 1, f_1 = 2$. Then $qR = Q$, $N_K(Q) = q^2$ and we say q is inert in K .

3) $g = 2, e_1 = e_2 = 1, f_1 = f_2 = 1$. Then $qR = Q_1 Q_2$,
 $N_K(Q_1) = N_K(Q_2) = q$, and we say q splits in K .

Now, before we proceed with our proof we need the following:

Lemma 4.4: i) $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$.

ii) If n is divisible by m , then F_n is divisible by F_m .

iii) If $m > 0$ and $p > 5$, then $(F_{mp}/F_m) > p$.

Proof: i) We shall carry out the proof by induction on m . $m = 1$,

$F_{n+1} = F_{n-1}F_1 + F_nF_2 = F_{n-1} + F_n$. Suppose it is true for $m = k$, and

$m = k + 1$: we shall prove it is true when $m = k + 2$. Thus let

$F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$, $F_{n+k+1} = F_{n-1}F_{k+1} + F_nF_{k+2}$. Adding the

last two equations term by term we obtain $F_{n+k+2} = F_{n-1}F_{k+2} + F_nF_{k+3}$.

And this is the required result.

ii) Let $n = mm_1$. We shall prove by induction on m_1 . For

$m_1 = 1$, $n = m$ obvious. Suppose F_{mm_1} is divisible by F_m , and

consider $F_{m(m_1+1)}$. But $F_{m(m_1+1)} = F_{mm_1+m}$ from (i), we get

$F_{m(m_1+1)} = F_{mm_1-1}F_m + F_{mm_1}F_{m+1}$. The right hand side of this equation is divisible by F_m . Then $F_{m(m_1+1)}$ is divisible by F_m .

iii) We shall prove by induction on m . If $m = 1$, then

$(F_{mp}/F_m) = F_p > p$. Suppose $m > 1$, and let $m = k + 1$, then

$$\begin{aligned} \frac{F_{(k+1)p}}{F_{k+1}} &= \frac{F_{kp+p}}{F_{k+1}} = \frac{F_{kp-1}F_p + F_{kp}F_{p+1}}{F_{k+1}} > \frac{F_{kp-1}F_p + F_{kp}F_p}{F_{k+1}} = \frac{(F_{kp-1} + F_{kp})F_p}{F_{k+1}} \\ &= \frac{F_{kp+1}F_p}{F_{k+1}}. \end{aligned}$$

But since $kp+1 > k+1$, $p > 5$, then $F_{kp+1} > F_{k+1}$ that is

$$\frac{F_{kp+1}}{F_{k+1}} > 1. \text{ Thus } \frac{F_{(k+1)p}}{F_{k+1}} > \frac{F_{kp+1}}{F_{k+1}} F_p > F_p > p. \text{ i.e. } \frac{F_{mp}}{F_m} > p,$$

$\forall m > 0, p > 5$.

We now go back to our proof. We have $R = \mathbb{Z}[\theta]$, $\theta^2 = \theta + 1$. The conjugate of θ in R is equal to $1 - \theta$, since it satisfies the same minimal polynomial of θ . i.e. we have two automorphisms (and only two) on R . One of them is the identity and the other is given by $\sigma: R \xrightarrow{\text{auto.}} R$, $\sigma(\theta) = 1 - \theta$.

Now we have the following:

$$\text{i)} \quad \theta^n = F_{n-1} + F_n \theta, \quad n > 0$$

$$\text{ii)} \quad F_n = \frac{\sigma\theta^n - \theta^n}{\sigma\theta - \theta}.$$

For (i), we shall prove by induction on n . $n = 1$, then

$$\theta = F_0 + F_1\theta = 0 + \theta = \theta. \text{ Suppose } \theta^k = F_{k-1} + F_k\theta. \text{ Then}$$

$$\theta^{k+1} = \theta^k\theta = (F_{k-1} + F_k\theta)\theta = F_{k-1}\theta + F_k\theta^2 = F_{k-1}\theta + F_k(\theta+1)$$

$$= F_k + (F_{k-1} + F_k)\theta = F_k + F_{k+1}\theta. \text{ To see (ii), } \sigma\theta^n - \theta^n =$$

$$= (F_{n-1} + F_n - F_n\theta) - (F_{n-1} + F_n\theta) = F_n(1-2\theta). \text{ Also } \sigma\theta - \theta = 1 - 2\theta.$$

$$\text{i.e. } \frac{\sigma\theta^n - \theta^n}{\sigma\theta - \theta} = F_n.$$

Now the proof will follow into two cases:

Case 1: Suppose $p \nmid F_m$ (p does not divide F_m). Since m divides mp , then by Lemma 4.4 (ii), F_m divides F_{mp} . i.e. (F_{mp}/F_m) is an integer, and by 4.4 (iii), it is greater than p . Hence there exists a prime q divides (F_{mp}/F_m) . Hence q divides F_{mp} , and we shall consider two cases:

i) Assume $q \nmid F_m$. Let $\epsilon = \frac{\sigma\theta}{\theta}$, ϵ is a unit of R , for its conjugate is $(\frac{\theta}{\sigma\theta})$, i.e. the norm of ϵ is $\frac{\sigma\theta}{\theta} \cdot \frac{\theta}{\sigma\theta} = 1$. Then $F_n = \frac{\sigma\theta^n - \theta^n}{\sigma\theta - \theta} = \theta^n \frac{\epsilon^n - 1}{\sigma\theta - \theta}$, and since $q \mid F_{mp}$ and $q \nmid F_m$, then $\epsilon^{mp} \equiv 1 \pmod{qR}$ and $\epsilon^m \not\equiv 1 \pmod{qR}$. Hence p divides the order of $\bar{\epsilon} = \epsilon + qR$, in $(R/qR)^*$, the set of unit elements of the finite F_q -Algebra R/qR , hence p divides the cardinality of $(R/qR)^*$, and before we proceed we need the following, which can be found in [10,p.125].

Lemma 4.5: If I is a non zero ideal of R , then the number of units of the finite ring R/I is given by

$$\text{card}(R/I)^* = N(I) \prod_{p \geq 1} \left(1 - \frac{1}{N(P)}\right).$$

$$\text{Now, } \text{card}(R/qR)^* = \begin{cases} (q-1)(q+1) & \text{if } qR = Q \\ q(q-1) & \text{if } qR = Q^2 \\ (q-1)^2 & \text{if } qR = Q_1Q_2. \end{cases}$$

And if p divides $q+1$, then p divides $\frac{1}{2}(q+1)$, since $q+1$ is even integer, (unless $q = 2$ when $p \mid q+1$, contradicts $p > 5$). But $\frac{1}{2}(q+1) < q$, then $p < q$. Similarly if p divides $q-1$, we get $p < q$. And if p divides q , then $p = q$, and in this case q ramifies in K and since 5 is the only prime ramifies in K then

$p = q = 5$, a contradiction with $p > 5$. So this q answers question (**).

ii) Assume $q \mid F_m$. And we have $q \mid \left(\frac{F_{mp}}{F_m} \right)$. Since $F_{mp} = \theta^{mp} \frac{\epsilon^{mp} - 1}{\sigma\theta - \theta}$,
 $F_m = \theta^m \frac{\epsilon^m - 1}{\sigma\theta - \theta}$ then $\frac{F_{mp}}{F_m} = \theta^{mp-m} \frac{\epsilon^{mp} - 1}{\epsilon^m - 1} = \theta^{mp-m} (1 + \epsilon^m + \dots + \epsilon^{m(p-1)})$.
 Since $q \mid \left(\frac{F_{mp}}{F_m} \right)$, then $\theta^{mp-m} (1 + \epsilon^m + \dots + \epsilon^{m(p-1)}) \equiv 0 \pmod{qR}$, and

since $q \mid F_m$, then $\epsilon^m \equiv 1 \pmod{qR}$. From the last two equations, we get $\theta^{mp-m} p \equiv 0 \pmod{qR}$. Hence $p = q$, and since $q \mid F_m$, then $p \mid F_m$ a contradiction.

Case 2: Suppose $p \mid F_m$ i.e. $\epsilon^m \equiv 1 \pmod{pR}$ since

$\frac{F_{mp}}{F_m} = \frac{\theta^{mp}}{\theta^m} \cdot \frac{\epsilon^{mp} - 1}{\epsilon^m - 1}$, so if we let $\epsilon^m = 1 + \delta$. Hence

$$\theta^{m-mp} \frac{F_{mp}}{F_m} = \frac{(\epsilon^m)^p - 1}{\epsilon^m - 1} = \frac{(1+\delta)^p - 1}{\delta} \quad \text{i.e.} \quad \theta^{m-mp} \frac{F_{mp}}{F_m} \\ = p + \binom{p}{2} \delta + \binom{p}{3} \delta^2 + \dots + \binom{p}{p-1} \delta^{p-1} \equiv p \pmod{p^2R}. \quad \text{Hence } p^2 \nmid \left(\frac{F_{mp}}{F_m} \right).$$

But we know $\frac{F_{mp}}{F_m} > p$, where $p > 5$. Then there exists a prime q

different from p , such that $q \mid (F_{mp}/F_m)$.

Now, if $q \mid F_m$, then by (ii), $q = p$ a contradiction to the fact that p is different from q . And if $q \nmid F_m$, then by (i), $q > p$, as required. This concludes the proof.

Conjecture 4.6: As a result of 4.3 and a discussion with A. Rhemtulla and A. Weiss we conjecture that G is in χ if and only if G is finitely generated nilpotent by finite group.

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